International Journal of Theoretical Physics, Vol. 45, No. 11, November 2006 (© 2006)
DOI: 10.1007/s10773-006-9172-6

Potential Symmetries and Associated Conservation Laws to Fokker-Planck and Burgers Equation

Jian-qin Mei[1,2] and Hong-qing Zhang[1]

Received December 22, 2005; accepted March 13, 2006
Published Online: May 25, 2006

Kara and Mahomed have derived an identity, which does not rely on use of a Lagrangian as needed to obtain conservation laws by Noethers theorem. By using the identity and symbolic computation, conservation laws arising from nonlocal symmetries are obtained for Fokker-Planck equation and burgers equation.

KEY WORDS: potential symmetries; conservation law; Fokker-Planck equation; burgers equation.

1. INTRODUCTION

The theory of continuous groups of transformations created by Sophus Lie has evolved to become one of the most important tools for geometric and algebraic study of general nonlinear partial differential equations. The generation of conservation laws of a system of differential equations from known ones using the symmetry properties of the system has been investigated over the years.

Anco and Bluman, Bluman *et al.* (1998) present a detailed discussion on the local nature of conservation laws derived from Noethers theorem. Nonlocal conservation laws are, by way of an identity, derived from nonlocal, canonical symmetries without reliance on a Lagrangian. Kara and Qu, Anco and Bluman, (1996) have shown that the generation of conservation laws of a system of differential equations from known ones using symmetry properties of the system can be successfully extended to nonlocal symmetries using an identity derived by Kara and Mahomed, Kara and Qu (2000). It was indicated that even though we obtained conserved vectors associated with nonlocal symmetries, it was much harder to obtain those associated with the so-called genuine potential symmetries.

[1] Department of Applied Mathematics, Dalian University of Technology, Dalian 116024 P. R. China.
[2] To whom correspondence should be addressed at; e-mail: meiqin-303@163.com.

In this article, we apply the identity derived in Kara and Qu, (2000) to construct (nonlocal) conservation laws from potential symmetries symmetries for Fokker-Planck equation and burgers equation.

2. POTENTIAL SYMMETRIES

Let us consider a partial differential equation of the form:

$$\triangle(x, t, u, u_x, u_t, u_{xx}, u_{tt}) = 0 \tag{1}$$

where x and t are independent variables and u is the dependent variable.

Bluman *et al.* (1998) Kara and Mahomed (2000) have introduced the concept of potential symmetry for any differential equation which can be written as a conservation law. In the case considered here, this means that (1) can be written as

$$D_x F(x, t, u, u_x, u_t) - D_t G(x, t, u, u_x, u_t) = 0 \tag{2}$$

where D_x and D_t are the total derivative operators. Introducing an auxiliary potential variable $v = v(x, t)$, it is possible to form the potential system, $S = 0$,

$$v_t = F, v_x = G \tag{3}$$

which is obviously equivalent to (2).

To compute the classical point symmetries of (3), we introduce the infinitesimal generator

$$X = \xi(x, t, u, v)\frac{\partial}{\partial x} + \tau(x, t, u, v)\frac{\partial}{\partial t} + \eta(x, t, u, v)\frac{\partial}{\partial u} + \varphi(x, t, u, v)\frac{\partial}{\partial v} \tag{4}$$

and its first-order prolongation

$$X^{(1)} = X + \eta^x \frac{\partial}{\partial u_x} + \eta^t \frac{\partial}{\partial u_t} + \varphi^x \frac{\partial}{\partial v_x} + \varphi^t \frac{\partial}{\partial v_t} \tag{5}$$

where

$$\eta^x = D_x\eta - u_x D_x\xi - u_t D_x\tau, \eta^t = D_t\eta - u_x D_t\xi - u_t D_t\tau$$

$$\varphi^x = D_x\varphi - v_x D_x\xi - v_t D_x\tau, \varphi^t = D_t\varphi - v_x D_t\xi - v_t D_t\tau$$

Considering the relation

$$X^{(1)} S|_{S^1} = 0$$

we obtain the defining equations of the classical point symmetries admitted by (3). Any admitted symmetry with infinitesimal generator X where ξ, τ or η depend on v is called potential symmetry of (2); potential symmetries are non-local symmetries.

3. DETERMINATION OF THE CONSERVATION LAW FROM POTENTIAL SYMMETRIES

We briefly outline the notation and give the necessary preliminaries used earlier. Let $x = (x_1, x_2, \ldots, x_n) \in R^n$ be the independent variables with coordinates x^i, and let $u = (u_1, u_2, \ldots, u_m) \in R^n$ be the dependent variables with coordinates u^α. The partial derivatives of u with respect to x are connected by the operator of total differentiation

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots$$

as $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha)$.

The collection of all first derivatives u_i^α will be denoted by $u_{(1)}$. Likewise, the collections of all higher order derivatives will be denoted by $u_{(2)}, u_{(3)}, \ldots$.

Consider an r-th order system of partial differential equations of n independent and m dependent variables, viz.,

$$E^\beta(x, u, u_{(1)}, \ldots, u_{(r)}), \beta = 1, \ldots, m \tag{6}$$

A conservation law of (6) is the equation

$$D_i T^i = 0, \tag{7}$$

on the solutions of (6). The tuple $T = (T^1, \ldots, T^n)$ is called a conserved vector of (6).

Suppose A is the universal space of differential functions. A Lie-Backlund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} + \ldots \tag{8}$$

where $\xi^i, \eta^\alpha \in A$ and the additional coefficients are

$$\zeta_i^\alpha = D_i(W^\alpha) + \xi^j u_{ij}^\alpha,$$

$$\zeta_{ij}^\alpha = D_j D_i(W^\alpha) + \xi^k u_{kij}^\alpha, \ldots \tag{9}$$

and W^α is the Lie characteristic function defined by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha \tag{10}$$

The following theorem and definition are recalled from Kara and Mahomed, Kara and Qu, (2000).

Theorem 3.1. *Suppose that X is a Lie-bäcklund symmetry of the system (1) such that the conserved form ω of (6), given by (7), is invariant under X. Then*

$$X(T^i) + D_j(\xi^j)T^i - T^j D_j(\xi^i) = 0 \tag{11}$$

Definition 3.1. A Lie-b$\ddot{a}$cklund symmetry X is said to be associated with a conserved vector T (or its corresponding conserved form ω) of the system (6) if X and T satisfy (11).

4. APPLICATIONS TO FOKKER-PLANCK EQUATION

Let us consider the Fokker-Planck Equations Pucci and Saccomandi, (1993); Saccomandi, (1997):

$$u_t = (xu_x)_x + u_{xx} \tag{12}$$

and the corresponding potential system

$$v_x = u,\, v_t = u_x + xu \tag{13}$$

It is shown in Pucci and Saccomandi (1993); Saccomandi (1997) a Lie-b$\ddot{a}$cklund symmetry generator of (13) is :

$$X = -x\frac{\partial}{\partial x} - \frac{\partial}{\partial t} + [u(x^2+2) + 2xv]\frac{\partial}{\partial u} + v(x^2+1)\frac{\partial}{\partial v} \tag{14}$$

is a (nonlocal) potential symmetry for (12).

We use X to construct an associated (nonlocal) conservation law for (12) by applying the identity (11) and the conserved form (7) to (12). Here Eq. (11) is the system:

$$-\frac{\partial T^1}{\partial x}x - \frac{\partial T^1}{\partial t} + (u(x^2+2) + 2xv)\frac{\partial T^1}{\partial u} + v(x^2+1)\frac{\partial T^1}{\partial v} - T^1 = 0 \tag{15}$$

$$-\frac{\partial T^2}{\partial x}x - \frac{\partial T^2}{\partial t} + (u(x^2+2) + 2xv)\frac{\partial T^2}{\partial u} + v(x^2+1)\frac{\partial T^2}{\partial v} = 0 \tag{16}$$

The conservation law (7) is

$$D_t T^1 + D_x T^2 = 0 \tag{17}$$

along the solutions of (13) so that

$$\frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} + (u_x + xu)\frac{\partial T^1}{\partial v} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} + u\frac{\partial T^2}{\partial v} = 0 \tag{18}$$

Separating by derivatives of u, we have

$$u_t : \frac{\partial T^1}{\partial u} = 0 \tag{19}$$

$$u_x : \frac{\partial T^1}{\partial v} + \frac{\partial T^2}{\partial u} = 0 \tag{20}$$

$$u : \frac{\partial T^1}{\partial v} x + \frac{\partial T^2}{\partial v} = 0 \tag{21}$$

$$: \frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} = 0 \tag{22}$$

First,we can solve the system (15)(16)(19) using Maple package and have

$$T^1 = \frac{F1(T, X)}{x}, T^2 = F2(T, X, U) \tag{23}$$

where $T = t - ln(x)$, $X = xv\exp(\frac{x^2}{2})$, $U = x^2(u + xv)\exp(\frac{x^2}{2})$.

Substituting this solution into (20)(21) one can get

$$\frac{\partial F1}{\partial X} + \frac{\partial F2}{\partial U} x^2 = 0 \tag{24}$$

$$\frac{\partial F1}{\partial X} + \frac{\partial F2}{\partial X} + \frac{\partial F2}{\partial U} x^2 = 0 \tag{25}$$

So $F2(T, X, U) = F3(T, U)$ and substituting these into (22),we can get

$$\frac{\partial F1}{\partial T}\frac{\partial F3}{\partial U} - \frac{\partial F3}{\partial T}\frac{\partial F3}{\partial U} - \frac{\partial F1}{\partial X}\frac{\partial F3}{\partial U}(U + X) + 2\frac{\partial F3}{\partial U} U^2 = 0$$

It is not difficult to solve this equation:

$$F3\,(T, U) = C_1 e^T U + P1\,(T) + P2\left(Ue^{2T}\right), \; F1\,(T, X) = C_1 e^T X + P1\,(T)$$

Thus the conservation laws T^1, T^2 associated with 'genuine' potential symmetry (14) have been derived.

$$T^1 = \frac{1}{x}(C_1 e^T X + P1\,(T)), T^2 = C_1 e^T U + P1\,(T) + P2(Ue^{2T}) \tag{26}$$

where $T = t - ln(x)$, $X = xv\exp(\frac{x^2}{2})$, $U = x^2(u + xv)\exp(\frac{x^2}{2})$.

5. APPLICATIONS TO BURGERS EQUATION

Let us consider the Burgers equation:

$$u_t = uu_x + u_{xx} \tag{27}$$

and the corresponding potential system

$$v_x = 2u,\, v_t = 2u_x + u^2 \tag{28}$$

According to the algorithm in Section 2 ,we can get the defining equations as follows :

$$\begin{aligned}
-2\tau_u = 0,\, -4\tau_v + 2\xi_u &= 0\\
2\phi_v - 4\tau_v u^2 - 2\eta_u - 2\tau_t + 2\xi_x &= 0\\
\phi_u - \tau_u u^2 - 2\tau_x - 2u\xi_u - 4\tau_v u &= 0\\
-2u\xi_u + 4\tau_v u + \phi_u + 2\tau_x - \tau_u u^2 &= 0\\
-2\eta + \phi_x - 2u\xi_x + 2\phi_v u - 4\xi_v u^2 - \tau_x u^2 - 2\tau_v u^3 &= 0\\
\phi_v u^2 - 2u^3\xi_v - 2\eta u - 4\eta_v u - \tau_v u^4 - 2\eta_x - 2u\xi_t - \tau_t u^2 + \phi_t &= 0
\end{aligned}$$

Solving this determining system yields

$$\begin{aligned}
\xi &= \frac{1}{2}(C_1 x + 2C_4)t + \frac{1}{2}C_2 x + C_5,\, \tau = \frac{1}{2}C_1 t^2 + C_2 t + C_3\\
\phi &= C_6 e^{c_1 t - \frac{v}{4}}(C_7 e^{\sqrt{c_1}x} + C_8 e^{-\sqrt{c_1}x}) + \frac{1}{2}(-2t - x^2)C_1 - 2C_4 x + C_9\\
\eta &= -\frac{1}{4}C_6 e^{c_1 t - \frac{v}{4}}[C_7\left(u - 2\sqrt{c_1}\right)e^{\sqrt{c_1}x} + C_8\left(2\sqrt{c_1} + u\right)e^{-\sqrt{c_1}x}]\\
&\quad -\frac{1}{2}(C_1 t + C_2)u - \frac{1}{2}C_1 x - C_4
\end{aligned}$$

So when $C_6 \neq 0$ and either C_7 or C_8 is nonzero, X is a (nonlocal) potential symmetry for (1).

We use X to construct an associated (nonlocal) conservation law for (27).

The conservation law (27) is

$$D_t T^1 + D_x T^2 = 0 \tag{29}$$

along the solutions of (28) so that

$$\frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} + (2u_x + u^2)\frac{\partial T^1}{\partial v} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} + 2u\frac{\partial T^2}{\partial v} = 0 \tag{30}$$

Separating by derivatives of u, we have

$$u_t : \frac{\partial T^1}{\partial u} = 0 \tag{31}$$

$$u_x : 2\frac{\partial T^1}{\partial v} + \frac{\partial T^2}{\partial u} = 0 \tag{32}$$

$$: \frac{\partial T^1}{\partial t} + \frac{\partial T^2}{\partial x} + u^2\frac{\partial T^1}{\partial v} + 2u\frac{\partial T^2}{\partial v} = 0 \tag{33}$$

Case 1:when $C_5 = a, C_3 = b, c_1 = C_1 = C_2 = C_4 = C_7 = C_9 = 0, C_8 = C_6 = 1$,

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial t} + \frac{ue^{\frac{v}{4}}}{4}\frac{\partial}{\partial u} + e^{\frac{v}{4}}\frac{\partial}{\partial v} \tag{34}$$

Equation (11) turns into this system:

$$a\frac{\partial T^1}{\partial x} + b\frac{\partial T^1}{\partial t} + \frac{ue^{\frac{v}{4}}}{4}\frac{\partial T^1}{\partial u} + e^{\frac{v}{4}}\frac{\partial T^1}{\partial v} = 0 \tag{35}$$

$$a\frac{\partial T^2}{\partial x} + b\frac{\partial T^2}{\partial t} + \frac{ue^{\frac{v}{4}}}{4}\frac{\partial T^2}{\partial u} + e^{\frac{v}{4}}\frac{\partial T^2}{\partial v} = 0 \tag{36}$$

We can solve the system (31)–(33)and (35), (36) using Maple package and have

$$T1 = \frac{1}{4}\left(C2 + C1\,e^{\frac{a\,(xb-a\,t)}{b^2}}\right)xa^{-1} + F\left(t - \frac{xb}{a}\right) + \left(C2 + C1\,e^{\frac{a\,x}{b}-t}\right)e^{-\frac{v}{4}}$$

$$T2 = \frac{1}{4ab}\Bigg[C1e^{\frac{a\,xb-a^2t}{b^2}}(2a\,(2\,a + b\,u)e^{-\frac{v}{4}} + (a\,x - b))$$
$$+ 2a(2aF\left(t - \frac{xb}{a}\right) + b(ue^{-\frac{v}{4}}C2 + 2\,C3))\Bigg]$$

6. CONCLUSION

It has been shown that one can generate a class of nontrivial conservation laws for some partial differential equations using some recent results and through Lie-backlund symmetry generator of the system. These conserved vectors are nonlocal as they are constructed from associated nonlocal symmetries of the partial differential equation. This method can be successfully extended to association with genuine nonlocal (potential) symmetries. It usually involves solving more difficult systems of PDEs. By with the help of symbolic computation,we have obtained the conservation laws T^1, T^2 associated with 'genuine' potential symmetry for Fokker-Planck equation and burgers equation .

ACKNOWLEDGMENTS

The work is supported by a National Key Basic Research Project of China (2004CB3180 00) and National Key Basic Research Development of China (Grant No. 1998030600).

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